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POSITIVELY CURVED n-MANIFOLDS IN R^{n+2}

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Introduction

In view of the difficulty of classifying *all* compact Riemannian manifolds with strictly positive sectional curvature, we make the additional hypothesis that the manifold is isometrically immersed in a Euclidean space with codimension 2. In § 1 we prove a theorem in what B. O'Neill has called "pointwise differential geometry" (i.e. linear algebra). This theorem is applied in § 2 to obtain results about the manifolds specified in the title. For instance, we show that a metric of positive curvature on $S^2 \times S^2$ cannot be induced by an immersion in R^6.

1. An algebraic theorem

Let V and W be real vector spaces of finite dimensions n and p respectively, and $B: V \times V \to W$ a symmetric bilinear form on V with values in W. Suppose $n \geq 2$ and W has an inner product $\langle\ ,\ \rangle$. Define the *associated curvature form* R_B: $\Lambda^2 V \times \Lambda^2 V \to R$ by

$$R_B(x \wedge y, z \wedge w) = \langle B(x, z), B(y, w)\rangle - \langle B(x, w), B(z, y)\rangle \ .$$

R_B is again symmetric, and is positive definite iff $R_B(\omega, \omega) > 0$ whenever $\omega \neq 0$. We say that R_B has *positive sectional values* iff $R_B(x \wedge y, x \wedge y) > 0$ whenever $x \wedge y \neq 0$.

Consider the following conditions on B:

(a) There exists an orthonormal basis $\{e_1, \cdots, e_p\}$ for W such that the real-valued forms on V defined by $(x, y) \mapsto \langle B(x, y), e_i\rangle$ are all positive definite.

(b) R_B is positive definite.

(c) R_B has positive sectional values.

Theorem 1. (a) $\Rightarrow$ (b) $\Rightarrow$ (c). *If* $p = 2$, *then* (c) $\Rightarrow$ (a). *In fact, let* $p = 2$ *and* $\mathscr{P} = \{B \mid R_B$ *has positive sectional values*$\}$. *Then there are continuous functions* e_1 *and* e_2 *from* $\mathscr{P}$ *to* W, *canonically determined by an orientation of* W, *such that for each* $B \in \mathscr{P}$, $\{e_1(B), e_2(B)\}$ *is an orthonormal frame for* W, *and the forms* $(x, y) \mapsto \langle B(x, y), e_i(B)\rangle$ *are both positive definite.*

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Proof. (a) $\Rightarrow$ (b): If B_i denotes the form $(x, y) \mapsto \langle B(x, y), e_i \rangle$, then $B(x, y) = \sum_i B_i(x, y)e_i$, and $R_B = \sum_i R_i$, where

$$R_i(x \wedge y, z \wedge w) = B_i(x, z)B_i(y, w) - B_i(x, w)B_i(z, y) .$$

To prove that R_B is positive definite, it suffices to prove that all the R_i are positive definite. For fixed i, let $\{x_1, \cdots, x_n\}$ be a basis for V which diagonalizes B_i; i.e., $B_i(x_j, x_k) = \lambda_j \delta_{jk}$. $\lambda_j > 0$ for all j, because B_i is positive definite. Then $\{x_j \wedge x_k \mid j < k\}$ forms a basis for $\Lambda^2 V$ which diagonalizes R_i with proper values $\lambda_j \lambda_k > 0$, so R_i is positive definite.

(b) $\Rightarrow$ (c) is trivial.

$p = 2$: Let R_B have positive sectional values. Then for all pairs (x, y) of linearly independent vectors,

$$\langle B(x, x), B(y, y) \rangle > \langle B(x, y), B(x, y) \rangle \geq 0 . \tag{1}$$

Since $n \geq 2$, $B(x, x) \neq 0$ when $x \neq 0$, and

$$\langle B(x, x), B(y, y) \rangle > 0 , \tag{2}$$

so long as x and y are both non-zero.

Now choose an element $x_0 \neq 0$ in V and an orientation for W. For $x \neq 0$ in V, let $\theta(x)$ denote the directed angle from $B(x_0, x_0)$ to $B(x, x)$. $\theta(x)$ is *a priori* defined only modulo 2π, but (2) implies that we can define θ as a continuous function from the non-zero elements of V to the interval $(-\pi, \pi)$. From the quadratic homogeneity of B, it follows that θ factors through the (compact) projective space of V, so it must attain its maximum $\theta_{\max}$ and minimum $\theta_{\min}$. Now (2) implies that

$$\theta_{\max} - \theta_{\min} < \pi/2 . \tag{3}$$

Let

$$\bar{\theta} = (\theta_{\max} + \theta_{\min})/2 , \tag{4}$$

$$\theta_1 = \bar{\theta} + \pi/4 , \tag{5}$$

$$\theta_2 = \bar{\theta} - \pi/4 . \tag{6}$$

Let $e_1(B)$ and $e_2(B)$ be the unit vectors in W such that the directed angle from $B(x_0, x_0)$ to $e_i(B)$ is θ_i. It is easy to see that $e_1(B)$ and $e_2(B)$ are independent of the choice of x_0 and that they depend continuously on $B \in \mathscr{P}$. (5) and (6) imply that $\{e_1(B), e_2(B)\}$ is an orthonormal frame. It follows from (3), (4), (5), and (6) that the angle between $B(x, x)$ and $e_i(B)$ is less than $\pi/2$ for any $x \neq 0$, so that the forms $(x, y) \mapsto \langle B(x, y)e_i(B) \rangle$ are both positive definite.

2. Applications

Let M^n be a Riemannian manifold isometrically immersed in Euclidean space $\boldsymbol{R}^{n+2}$. The Gauss equations state that the curvature tensor of M^n, considered as a symmetric bilinear form on tangent bivectors, is equal to R_B, where B is the second fundamental form of M^n, considered as a symmetric bilinear form on the tangent space with values in the normal space. The following result follows immediately from Theorem 1.

Theorem 2. *If M^n is a manifold of strictly positive sectional curvature, isometrically immersed in $\boldsymbol{R}^{n+2}$, then the curvature tensor of M^n is positive definite. If M^n is orientable, the normal bundle of M^n has a canonical trivialization, so M^n is stably parallelizable.*

Theorem 3. *Let M^n be a compact manifold of strictly sectional positive curvature, isometrically immersed in $\boldsymbol{R}^{n+2}$.*

(1) *Then $H^2(M^n, \boldsymbol{R}) = 0$.*

(2) *If n is even, the Euler characteristic of M^n is positive.*

(3) *If M^n is orientable, the Pontryagin and Stiefel-Whitney classes of M^n are trivial.*

Proof.

(1) By Theorem 1, the curvature tensor of M^n is positive definite. According to Berger [1], this implies that every harmonic 2-form on M^n vanishes identically.

(2) By taking the orientable double covering of M^n, if necessary, we may assume that M^n is orientable. Now the Gauss-Bonnet integrand of M^n, whose integral is the Euler characteristic, is positive when the curvature tensor is positive definite. (This last assertion is due to B. Kostant (unpublished)).

(3) M is stably parallelizable.

Remark. The product $S^m \times S^n$ $(m, n \geq 1)$ of two spheres is naturally embedded in $\boldsymbol{R}^{m+n+2}$ with non-negative sectional curvature. Theorem 3 implies that there is no immersion of $S^m \times S^n$ in $\boldsymbol{R}^{m+n+2}$ with positive sectional curvature, unless, perhaps, m and n are both greater than 2 and not both odd. (The case where m or n equals 1 is eliminated by the theorem of Bochner and Myers which states that the first Betti number (over $\boldsymbol{R}$) of a compact manifold of positive Ricci curvature must be zero.)

Problems. Classify all positively curved compact M^n isometrically immersed in $\boldsymbol{R}^{n+2}$. In case $n = 4$, Theorem 3 and the theorem of Bochner and Myers imply that M^4 must be a real homology sphere. If M^n is orientable and *embedded*, Theorem 2 and the Pontryagin-Thom construction [2, § 7] associate to M^n an element of $\pi_{n+2}(S^2)$. Is this element always zero (i.e., is M^n always framed cobordant to the unit sphere in a hyperplane of $\boldsymbol{R}^{n+2}$)?

In $\boldsymbol{R}^{n+1}$, a positively curved M^n has positive definite second fundamental form, and this leads to the result that M^n is the boundary of a convex body. In $\boldsymbol{R}^{n+2}$, we know by Theorem 1 that there is a quadrant in each normal space which contains the range of the second fundamental form. Perhaps this fact can be used to obtain global results concerning the way in which M^n lies in $\boldsymbol{R}^{n+2}$.

A restricted version of the problem above is to classify all positively curved compact n-dimensional manifolds isometrically immersed in $S^{n+1} \subseteq \boldsymbol{R}^{n+2}$.

Bibliography

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